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Wave reflexion from beaches

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The usual way of posing the problem for the reflexion of wave trains from beaches seems inevitably to imply perfect reflexion. Energy considerations show that wave absorption must be associated with the degradation of mechanical energy either through wave breaking or viscous effects. Some experiments reported here showed substantial wave absorption in the absence of any breaking.

We describe some theoretical and experimental work aimed at assessing the role played by friction at the bottom in determining the reflexion coefficient of a beach. The results suggest that, if the parameter $(\nu\omega^3)^{\frac{1}{2}}/g\alpha^2$ is not too small, bottom friction can be a significant factor in the absorption process for waves on beaches. Here ν represents the kinematic viscosity (or perhaps an 'eddy' viscosity) of the fluid, ω is the frequency of the motions, α is the slope of the beach and g is the acceleration due to gravity.

1. Introduction

Beaches have been used in hydraulics laboratories for over a century as the primary means of absorbing wave energy (see, for example, Stoker 1957; Meyer & Taylor 1972). It appears to be generally agreed that the main mechanism for absorbing the wave energy is the breaking that occurs over most beaches, converting the wave energy into heat or into circulations in the water (the undertow). However, in practice it is difficult to test this conjecture for the following reasons. It is not possible to make a direct determination of the energy absorption, so this has to be inferred from observations of the wave field remote from the beach, and such inferences are usually complicated by the nonlinear nature of waves in hydraulics experiments. Also, there are no adequate theoretical models to describe the physics of breaking waves on beaches.

Nevertheless we were surprised to observe, in the course of another experiment, that, even when no breaking occurred, the beach reflected only a small fraction of the incident waves and this led us to reappraise what is known about wave absorption. An example of the experimental observations is given in figure 1, where the wave profiles observed in one of our experiments are shown at a number of stations. (We also confirmed by direct visual observation that no wave breaking occurred over the beach.) In the far field, where the wave amplitudes were small enough to justify the use of linear calculations, it was found that the reflected wave component was only about 10 % of the amplitude of the incident wave. So it would appear that the property

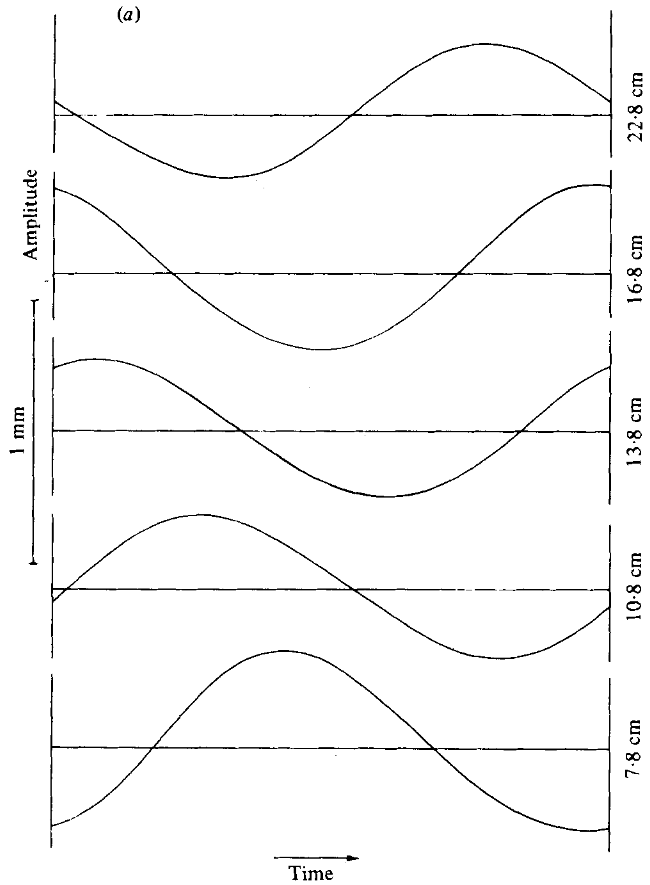

FIGURE 1 (a). For the caption see facing page.

of wave breaking is not necessarily fundamental to the mechanism of wave absorption on beaches.

In view of this result we would like to consider the following question. Suppose that the wave motion far from the beach has a given frequency and comprises a periodic wave train incident normally on the beach, together with a reflected train. What, then, is the relative magnitude of the incident and the reflected wave components, assuming that the waves do not break?

A discussion of the usual rationale for modelling waves on beaches is given in Stoker's (1957) book. Once the frequency of the waves is fixed, the irrotational flow over a plane sloping beach yields two different kinds of standing-wave solution to the linear wave problem. One of the solutions has finite amplitude, the other infinite amplitude, at the shoreline. However, far from the shoreline, these two solutions have a phase difference of $\pi/2$ and thus a suitable combination of them can be used to describe an arbitrary simple-harmonic progressive wave. The fact that one of these solutions is singular at the shoreline does not matter, it is argued, because in any case the waves break before they reach the shore and this precludes the applicability of the theory in that region. Since the specification of the amplitude of the progressive

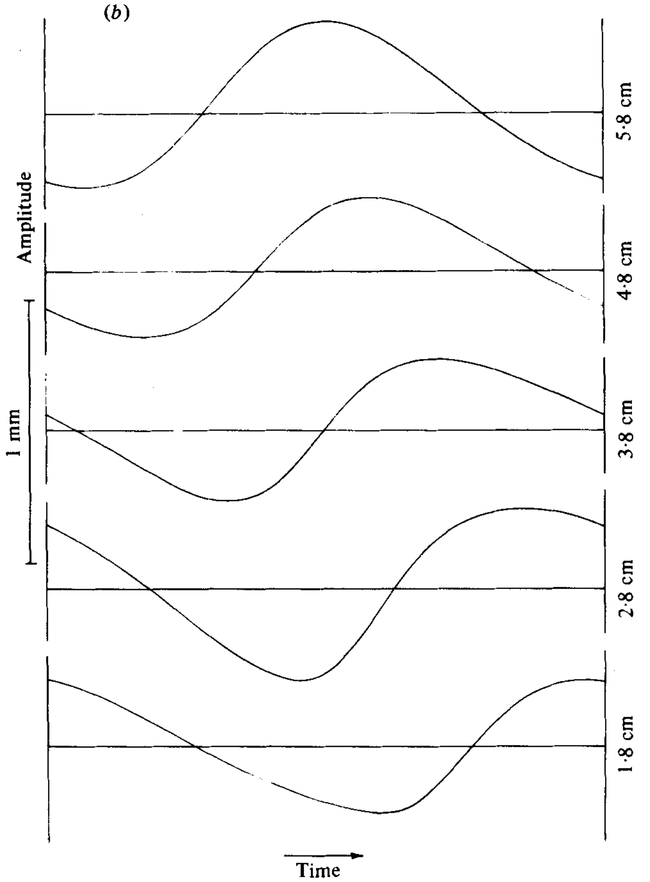

FIGURE 1. The observed wave profiles at various distances from the mean shoreline for (time) periodic waves incident on a beach of slope 1:11·0. The period of the waves was 0·6930 s. Note that the horizontal lines shown in this figure do not necessarily represent the undisturbed level of the water.

wave fixes uniquely the 'strength' of the singular solution, this procedure can, in principle, be used to describe the wave amplitudes over the beach (in the region beyond the breaking zone) when the magnitude of the reflected component is known. Whether or not this model gives a reasonable description of the wave amplitudes over a plane beach has not, to our knowledge, been carefully checked in the laboratory.

If, on the other hand, wave breaking does not occur, the assumption that the flow is irrotational would, at first sight, appear to be applicable. Then, if the motion in the far field can be described by simple harmonic waves at a given frequency, the conservation of mass and energy imply that the reflexion coefficient must be unity, irrespective of the wave amplitudes over the beach.

Arguments of this kind led us to the conclusion that the wave absorption in the above experiment must have been the result of the degradation of mechanical energy through frictional effects on the beach, this being the most important type of viscous action. So, as a first step, we decided to investigate the effects of bottom friction over the beach when the wave amplitudes are so small that nonlinearities, deriving from the free-surface condition, can be neglected. Under these conditions a standard boundary-layer calculation indicates that the effects of bottom friction are likely to be important in determining the wave field over the beach. An assessment of the experimental results in the light of these calculations suggests that the same kind of mechanism could apply on the field scale.

An important aspect of the problem, especially on the laboratory scale, is the way in which the free surface attaches to the beach. In order to vary this condition and to investigate the sensitivity of the absorption process to 'inshore' effects we have examined the situation in which the beach was terminated by a vertical cliff rather than the usual shoreline. The theory is also compared with a number of previous experimental investigations.

The experiments indicated that the amplitude of the reflected-wave component was reduced significantly if a small gap was left between the sides of the beach and the walls of the channel. A brief description is also given of a property of the flow in the boundary layer not far from the shoreline.

2. Energy considerations

It can be shown that the property of perfect wave reflexion, manifested by many of the models for a periodic wave train incident on a beach, is a direct consequence of the assumed form for the far field and the conservation of mechanical energy. In determining the appropriate form for the far field it should be recalled that considerations of mass flux in waves (see Phillips 1966) indicate that there must be a second-order mass outflow from the beach if the incident wave is not perfectly reflected. This is usually neglected (sensibly) in linearized theories where the far field might therefore be expected to be described in terms of incident and reflected wave trains. This suggests working on the basis of the following assumptions: (1) The flow is irrotational. (2) The surface elevation is a single-valued function of position and time, and this function is 'smooth'. (3) At large distances from the shore the wave amplitudes are sufficiently small to justify the use of linear theory. It is further assumed that the far field consists of an incident periodic wave train, together with a reflected wave train of unknown amplitude and phase, but having the same frequency as the incident train. (4) The flow regime is a stationary, quasi-periodic function of time. (5) Surface tension is negligible. (6) The motions are independent of the distance along the shore.

Given these conditions it follows that *the reflexion coefficient is unity.* A proof of this result, based on an energy argument, was included in the original version of the paper and is given in Mahony & Pritchard (1978).†

Note that assumptions (1) and (2) preclude the possibility of breaking waves. However, there is no requirement that the wave amplitude over the beach should be small and indeed there is nothing in the proof that in any way further limits the amplitudes reached as the waves approach the shore.

On the other hand, there may be inconsistencies between the assumed representation of the far field (assumption 3) and large inshore amplitudes. Should the waves grow to large amplitude near the beach (this need not necessarily happen) nonlinear effects will produce higher harmonics in the temporal field and consequently higher-

† These reports are available from the National Lending Library.

frequency waves might be expected in the reflected wave field. Under these conditions our theorem merely ascertains that the incident energy flux balances the reflected energy flux in all the reflected modes. (In the experiments described below we did not observe appreciable levels of such harmonics in the far field.) Also, larger amplitude waves will have an associated second-order mass flux towards the shore, so there must be a corresponding outflow. With perfect reflexion this is realized through the reflected wave. But if there is wave absorption then, within the framework of potential theory, the additional outflow must be distributed throughout the depth (cf. Phillips 1966.)

If the effects of surface tension are included, there appears a term deriving from the rate of working of the surface-tension forces at the shoreline. There is no obvious physical reason that enables us to determine the sign of this term and so the possibility arises that the reflexion coefficient could exceed unity. (It is apparent from experimental results, as indicated below, that surface-tension effects can be important on the laboratory scale, but in geophysical contexts it is inconceivable that this could be the case.)

3. Linear boundary-layer theory

If there is a periodic to-and-fro motion of a fluid above a rigid boundary, the thickness of the Stokes boundary layer is† $O(\nu/\omega)^{\frac{1}{2}}$. We shall assume that this thickness is small compared with the depth of the fluid and with the radius of curvature of the bottom, but note that the first of these assumptions must break down very near the shoreline. It is also assumed that the inertial terms in the momentum equation may be neglected: this can be justified *a posteriori* if the slope of the boundary is small everywhere. Then, with regard to axes (s, n) parallel to and normal to the boundary, the tangential momentum equation is given approximately by

$$\partial u/\partial t = \nu\, \partial^2 u/\partial n^2,$$

where u is the velocity in the s direction. The solution of this equation satisfying the no-slip condition at the rigid boundary is

$$u = U(s)\, e^{-i\omega t}\left\{1 - \exp\left[-(\omega/\nu)^{\frac{1}{2}}\left(\frac{1-i}{\sqrt{2}}\right) n\right]\right\},$$

in which $U(s)$ is the tangential velocity just 'outside' the boundary layer and ω is the angular frequency of the motion. For such a velocity distribution it follows from the continuity equation that there must be a normal velocity $V(s)\, e^{-i\omega t}$ 'outside' the boundary layer, namely

$$V(s) = (\nu/\omega)^{\frac{1}{2}}\, e^{i\pi/4}\, dU/ds.$$

Thus, following the standard procedure, we find that the corrected boundary condition for the velocity potential at the bottom is

$$\partial\phi/\partial n = -(\nu/\omega)^{\frac{1}{2}}\, e^{i\pi/4}\, \partial^2\phi/\partial s^2. \tag{3.1}$$

† For the experiments described below the quantity $(\nu/\omega)^{\frac{1}{2}} \approx 0{\cdot}3$ mm whereas, for 10-second waves and a value of ν of 10 $\mathrm{cm^2\,s^{-1}}$ (as one might expect for ocean beaches), we have that $(\nu/\omega)^{\frac{1}{2}} \approx 4$ cm. Some estimates of an 'eddy' viscosity, under roughly these conditions, are given in Jonsson & Carlsen (1976).

This may be interpreted physically as correcting the no-flux condition at the rigid boundary for the effects of the displacement thickness of the Stokes boundary layer.

If we further suppose that the waves are of small enough amplitude for the non-linear part of the free-surface condition to be neglected,† the resultant mathematical problem at hand is to find the complex-valued function ϕ (where $\mathrm{Re}\,(\phi \exp(-i\omega t))$ is the velocity potential) such that

$$\Delta\phi = 0, \tag{3.2a}$$

and

$$\phi_z - (\omega^2/g)\,\phi = 0 \quad \text{on} \quad z = 0, \tag{3.2b}$$

together with (3.1) on $z = -b(x)$. Here x and z respectively represent horizontal and vertical coordinates with the origin at the shoreline.

The asymptotic structure of the waves for large x cannot be taken in the same form as that employed in § 2 because, owing to the viscous attenuation, the wave could not have travelled from infinity if it felt the bottom everywhere. So, for the present, we shall assume that $b \to \infty$ sufficiently rapidly with x that ϕ can be written in the form

$$\phi \sim (e^{-ikx} + r\, e^{ikx})\, e^{kz}$$

for x large and positive. In this expression r is a complex number and the wavenumber k is given by

$$k = \omega^2/g. \tag{3.3}$$

Let us now consider an energy argument of the kind used in the proof of the result quoted in § 2. Starting from the identity (cf. Fitzgerald 1976)

$$\mathrm{Im} \int_D \phi^* \Delta\phi = 0,$$

where * denotes the complex conjugate and D is the region occupied by the fluid, we find that

$$rr^* = 1 - (2\nu/\omega)^{\frac{1}{2}} \int_0^\infty \left|\frac{\partial\phi}{\partial s}\right| ds. \tag{3.4}$$

The integral here is evaluated along the bottom.

A natural length scale for the waves is given by the relation (3.3). On this scale the contribution per unit length to the damping of the waves is $O\{(2\nu\omega^3)^{\frac{1}{2}}/g\}$ which is usually small. But, for a beach typically of slope α, the length scale of waves significantly influenced by the bottom is $O(g\alpha/\omega^2)$. Moreover, the damping occurs over a distance of $O(1/\alpha)$ so a better measure of the damping effects is likely to be given by the parameter $(2\nu\omega^3)^{\frac{1}{2}}/g\alpha^2$. It is therefore apparent that a more detailed calculation should be made.

† It would not be difficult to modify this theory to include a small nonlinear correction. The effects of surface tension can also be included. On the other hand, it would be more difficult to account for the boundary-layer structure in a consistent manner at this level (see, for example, Dore 1977).

4. Linear shallow-water theory

If we consider only that zone of the beach for which the shallow-water approximations can be used it is possible to develop a simple analytic theory.† Let us suppose that the slope $b'(x)$ of the bottom is small everywhere, being typified by a parameter α. It follows from (3.2), for dynamics permitting a wave-like response, that the horizontal scale L of the motions is

$$L \sim \alpha g/\omega^2.$$

The corresponding vertical length scale is αL. These suggest the introduction of scaled coordinates (X, Z) defined by

$$X = x/g\alpha\omega^{-2}, \quad Z = z/g\alpha^2\omega^{-2}, \tag{4.1}$$

and let us suppose that the bottom is now specified by $Z = -\beta(X)$, where the scalings have been chosen so that β' is of unit order. Equations (3.2) and (3.1) are then transformed respectively to

$$\phi_{ZZ} + \alpha^2\phi_{XX} = 0, \tag{4.2a}$$

$$\phi_Z - \alpha^2\phi = 0 \quad \text{on} \quad Z = 0, \tag{4.2b}$$

and

$$\phi_Z + \alpha^2\beta'\phi_X = (\nu\omega^3/g^2)^{\frac{1}{2}}\, e^{i\pi/4}[(1-\alpha^2\beta'^2)\,\phi_{XX} - 2\beta'\phi_{XZ}]\,(1+\alpha^2\beta')^{-\frac{1}{2}}$$
$$\text{on} \quad Z = -\beta(X). \tag{4.2c}$$

If we assume a power-series expansion in Z for ϕ, it follows from (4.2a) that ϕ can be written in the form

$$\phi = \Phi_0(X) + \Phi_1(X)\,Z - \alpha^2\{\tfrac{1}{2}\Phi_0''Z^2 + \tfrac{1}{6}\Phi_1''Z^3\} + O(\alpha^4),$$

where Φ_0, Φ_1 may also include terms depending on α, but we do not intend to retain terms $O(\alpha^4)$. The boundary condition (4.2b) at the free surface indicates that

$$\Phi_1 = \alpha^2\Phi_0 + O(\alpha^4),$$

so that

$$\phi = \Phi_0(X)\,\{1+\alpha^2 Z\} - \tfrac{1}{2}\alpha^2\Phi_0''Z^2 + O(\alpha^4).$$

Thus, the boundary condition (4.2c) at the bottom leads to an equation for Φ_0 namely

$$\Phi_0 + \Phi_0''\beta(X) + \beta'(X)\,\Phi_0' = [(\nu\omega^3)^{\frac{1}{2}}/g\alpha^2]\,e^{i\pi/4}\Phi_0'' + O(\alpha^2). \tag{4.3}$$

In this equation the effect of the boundary layer on the bottom, as expressed by the term on the right-hand side, is enhanced by the factor α^{-2}. So, although the dimensionless number $(\nu\omega^3)^{\frac{1}{2}}/g$ might suggest very small viscous effects, it would appear from (4.3) that the viscous effects can play an important role on beaches of small slope.

If we set $[(\nu\omega^3)^{\frac{1}{2}}/g\alpha^2]\,e^{i\pi/4} = \gamma$ then (4.3) can be written as

$$\frac{d}{dX}\left\{[\beta(X)-\gamma]\,\frac{d\Phi_0}{dX}\right\} + \Phi_0 = 0, \tag{4.4}$$

† When shallow-water theory is not applicable everywhere the problem can be formulated in terms of a fairly straightforward integral equation. However, we have not pursued this line here because the results indicate that virtually all the damping occurs in zones for which the shallow-water approximation is a good one.

which, for the particular case of a plane beach, reduces to a variant of Bessel's equation, namely

$$\frac{d}{dX}\left\{(X-\gamma)\frac{d\Phi_0}{dX}\right\}+\Phi_0 = 0. \tag{4.5}$$

Define a new variable ξ given by

$$\xi = 2(X-\gamma)^{\frac{1}{2}},$$

with the branch defined by the requirement that ξ should have a large positive real part when X is large and positive. Then (4.5) becomes

$$\frac{d^2\Phi_0}{d\xi^2}+\frac{1}{\xi}\frac{d\Phi_0}{d\xi}+\Phi_0 = 0, \tag{4.6}$$

and this is Bessel's equation of zero order.

4.1. *Some limitations*

Our main concern in this paper is to examine how well the basic equation (4.4) describes the experimental situation for periodic wave trains incident on beaches. Equation (4.4) can be interpreted in terms of a reduced effective depth arising from a complex displacement thickness of the bottom boundary layer. However, in any practical realization of such flows there are a number of other effects that may be important, especially in the zone near the shore, and these can complicate the choice of appropriate boundary conditions for the model.

In the laboratory, for example, the effects of surface tension near the shoreline and the manner in which the shoreline is established can have a bearing on the results (cf. § 6). If the surface slopes are small everywhere, the theory leading to (4.4) can be modified to allow for surface tension. This leads to the equation

$$\mu[(\gamma-\beta)\,\Phi_0^{\text{iv}}-\beta'\Phi_0'''] + (\beta-\gamma)\,\Phi_0''-\beta\Phi_0'+\Phi_0 = 0, \tag{4.7}$$

where $\mu = T\omega^4/\rho g^3\alpha^2$, with T representing the surface tension. In hydraulics experiments μ is usually quite small, typically of the order of 0·01 or less, though this is not always the case (e.g. see some experiments made by Feir 1966, where μ took values as large as 0·5). For small values of μ, (4.7) suggests that surface-tension effects will be directly significant only in a zone close to the water line of dimensionless extent $O(\mu^{\frac{1}{2}})$.

In practice the wave amplitudes are not infinitesimally small, so nonlinear effects might be important. These effects could be manifested either by wave breaking or through the generation of higher harmonics leading to a change in wave profile close inshore. Associated with the former of these will be an energy loss from the periodic wave structure, with a consequent reduction in the reflected component; the latter is probably of less importance. For example, at second order in the amplitude of the incident wave, we can expect mean flows and second-harmonic flows to be generated by the incident wave field. It is only at third order that these terms contribute to the balance of a flow-field component with the frequency of the incident wave. Therefore, a rough estimate of the importance of these effects may be obtained from the change in depth arising from the second-order mass flux associated with the incident wave. This change is $O(a^2\omega^2/g)$, where a is the wave amplitude, (see Phillips 1966, p. 55) which suggests that the present theory should not be used when $a\omega^2/g \sim \alpha x$, or $X \sim a^2\omega^4/g^2\alpha^2$. (In the present experiments this length scale was about 10^{-4}; on ocean beaches it might typically be 10.)

The model itself derives from the assumption that the boundary layer on the bottom and the potential flow above it can be considered effectively as separate zones. This restricts the validity of the model at most to regions where the displacement thickness does not penetrate the surface. Therefore, the use of (4.4) is not justified† when X is as small as $|\gamma|$.

The inevitable conclusion from these considerations is that the model equation (4.4) should not be used in a zone near $X = 0$. However, there are a number of possibilities as to which are the dominant effects to be included near the shore. Indeed, it is likely that no single inshore boundary condition will be entirely appropriate for (4.4) for the range of experimental results available.

In addition, some care is needed in deriving a boundary condition at large distances from the shore. If the beach shelved indefinitely, the incoming waves would feel the bottom at a depth $O(g/\omega^2)$, which corresponds to a value of X of order α^{-2}. In the experiments to be described the extent of the beach was much less than this, so we have specified a value, X_T, at the toe of the beach, at which the amplitude of the incident wave is to be defined.

4.2. *Some properties of the 'off-shore' zone*

On typical ocean beaches the parameter $|\gamma|$ can be quite large (values of 16 seem not to be unreasonable, cf. § 6) so that we can make use of asymptotic solutions to examine the properties of the waves. Solutions of (4.6) take the form

$$\Phi_0(\xi) = AH_0^{(1)}(\xi) + BH_0^{(2)}(\xi), \tag{4.8}$$

where $H_0^{(1)}, H_0^{(2)}$ are the Hankel functions and A, B are constants. For large positive values of $\operatorname{Re}(\xi)$ the Hankel functions admit asymptotic expansions of the form (see Abramowitz & Stegun 1965)

$$\Phi_0(\xi) \sim \left(\frac{2}{\pi\xi}\right)^{\frac{1}{2}} \{A\, e^{i(\xi-\frac{1}{4}\pi)} + B\, e^{-i(\xi-\frac{1}{4}\pi)}\}, \tag{4.9}$$

from which we see that the second term of (4.8) represents the incident wave and the first term the reflected wave. If we write $\xi(= 2[(X-\gamma_0)-i\gamma_0]^{\frac{1}{2}})$ in the form

$$2\rho^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta - i\sin\tfrac{1}{2}\theta),$$

where $\gamma_0 = |\gamma|/\sqrt{2}$, $\rho = [(X-\gamma_0)^2+\gamma_0^2]^{\frac{1}{2}}$ and $\theta = \arctan[\gamma_0/(X-\gamma_0)]$, it is apparent when $|\gamma|$ is large that the asymptotic form (4.9) provides a good representation of (4.8) over the entire domain. In particular, when $X \gg \gamma_0$, the amplitude of the incident wave varies in space approximately as $X^{-\frac{1}{4}}\exp(-\gamma_0/X^{\frac{1}{2}})$. Thus, as the wave runs up the beach its amplitude initially grows like $X^{-\frac{1}{4}}$, as expected from inviscid theory.

† In this context Sir James Lighthill (private communication) has pointed out to us the need to include an allowance for the effect of the boundary layer on the surface boundary condition. In a particular case, namely the reflexion of waves from a vertical barrier in deep water, he has shown that this effect can be significant, especially for waves whose wavelength is of the order of or smaller than that of the boundary-layer thickness.

But eventually the exponential term limits the wave amplitude to a maximum value at $X = 2|\gamma|^2$ (of approximately $0{\cdot}51(\alpha|\gamma|)^{-\frac{1}{2}}$ times the amplitude at $X = \alpha^{-2}$, when $\alpha|\gamma|$ is small). The amplitude then decays through the dominant effect of the exponential term (taking a value at $X = |\gamma|$ of approximately $2{\cdot}24\, e^{-0{\cdot}43|\gamma|}$ times that at the maximum). It is therefore apparent, for large values of $|\gamma|$, that the incident wave will have been severely attenuated before it reaches the 'inshore' zone $X \sim |\gamma|$, where the problem needs to be reformulated. This 'inshore' zone appears as a $(1/|\gamma|)$-neighbourhood of the shore, when viewing the flow with an X-scale of $O(|\gamma|^2)$, suggesting the use of an inner and outer formulation.

For this reason we decided to examine the structure of the eigensolutions for the potential flow in a wedge, satisfying the free-surface condition at the upper surface and the condition (3.1) at the bottom. We seek solutions in terms of polar co-ordinates (σ, θ) with the origin at the shoreline, of the form

$$\phi_\lambda = \sigma^\lambda\{F_0(\theta,\lambda) + \sigma F_1(\theta,\lambda) + \ldots\},$$

where λ is a real parameter and $F_0, F_1, \ldots$ are real functions. Then F_0 is of the form

$$F_0 = \cos\lambda\theta, \quad \lambda = \pm(m+\tfrac{1}{2})\pi/\alpha, \quad m \in \mathbb{N}.$$

Since α typically is small, such large values for λ indicate that solutions involving negative powers of σ would be quite large, even at moderate values of σ, whereas those involving positive powers would be small. Presumably the former kind of solution could be ruled out on the basis of observation, with the latter suggesting that ϕ should be small in the 'inshore' zone.

In the laboratory the values of $|\gamma|$ usually range between about 0·25 and 1 or 2. Nevertheless, the features described above for large values of $|\gamma|$ seem broadly to carry over to the smaller values. For example, the amplitude of the incident wave, which is proportional to $|BH_0^{(2)}(\xi)|$ is shown in figure 2 for various values of $|\gamma|$. When compared with the case $\gamma = 0$, the graphs show how the influence of the boundary layer eventually overcomes the shelving effect of the beach, but that the influence of the boundary layer is concentrated nearer the shore as $|\gamma|$ decreases.

4.3. *Boundary conditions*

Suppose, as indicated above, we refer to the wave amplitudes at the point $X = X_T$, corresponding to the toe of the beach, to determine the reflexion coefficient. Let $\xi_T = 2(X_T - \gamma)^{\frac{1}{2}}$ and suppose that X_T is large enough for the asymptotic representation to be utilized. Then it follows that, at $X = X_T$, the ratio of the amplitude of the reflected wave train to that of the incident wave train is

$$|A|\exp[|\gamma|/(2X_T)^{\frac{1}{2}}]/|B|\exp[-|\gamma|/(2X_T)^{\frac{1}{2}}].$$

Thus, the reflexion coefficient, when determined by amplitudes at the toe of the beach, is given by

$$r(\gamma) = |A/B|\exp(|\gamma|(2/X_T)^{\frac{1}{2}}), \tag{4.10}$$

where $|A/B|$ is to be determined from the 'inner' boundary conditions on ϕ.

In view of the complicated physical situation near the shore we decided not to attempt a detailed calculation of the 'inner' boundary condition. But, guided by the

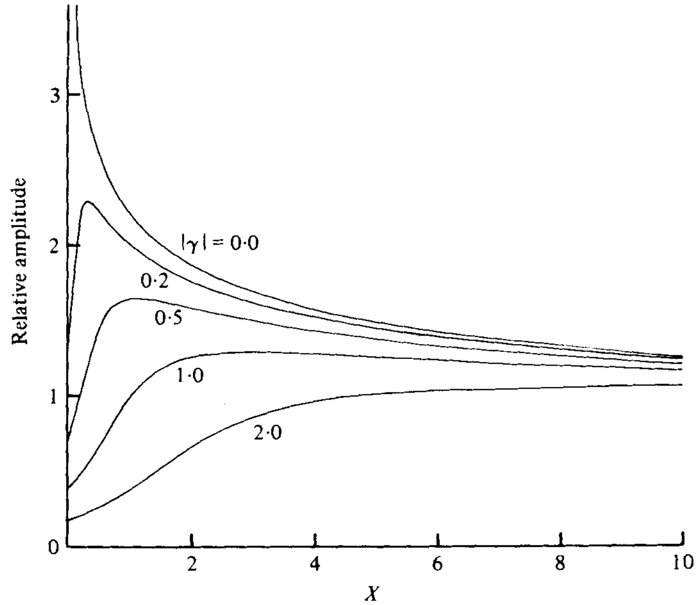

FIGURE 2. The amplitude according to (4.8) of the incident wave, relative to its amplitude at $X = 25{\cdot}0$, for various values of $|\gamma|$.

observation in our experiments that there was very little movement at the shoreline and by the preceding discussion, we have fixed on the condition

$$\phi(0) = 0. \tag{4.11}$$

While the choice of (4.11) was influenced by observations from our own experiments, we recognize that this boundary condition might be quite inappropriate when comparisons are to be made with other experiments. (One situation in which we would anticipate significant errors from this source is when $|\gamma|$ is fairly small and very little energy is absorbed in the off-shore zone. Then the wave amplitudes can become quite large leading to substantial movement at the shoreline or to wave breaking.) More generally, the inshore zone might be expected to provide an 'effective' origin, Y, for the outer flow, such that $\phi(Y) = 0$. To examine how changes in Y might affect the reflexion coefficient, let us suppose that Y is real and lies in the interval $[-|\gamma|, |\gamma|]$. Then, if we write $\xi_Y = 2(Y-\gamma)^{\frac{1}{2}}$, we have that

$$r_Y(\gamma) = \exp\left(|\gamma|(2/X_T)^{\frac{1}{2}}\right)\left|\frac{H_0^{(2)}(\xi_Y)}{H_0^{(1)}(\xi_Y)}\right|. \tag{4.12}$$

A graph of the reflexion coefficients thus obtained, for $Y = -|\gamma|, 0, |\gamma|$, and with X_T infinite, is given in figure 3(*a*). It may be seen from the graph that the reflexion coefficient is not especially sensitive to variations in Y: for example, with $|\gamma| = 0{\cdot}5$, about $0{\cdot}5$ % of the energy would be reflected if the relevant boundary condition were $\phi(0) = 0$, and less than 5 % of the energy would be reflected if we took $\phi(|\gamma|) = 0$. Moreover, the curve $r_{|\gamma|}$ is likely to be an overestimate of any shift induced by the inshore zone, since the condition $\phi(|\gamma|) = 0$ would seem to imply considerable constraints on the flow field.

Although the reflexion coefficient might not be particularly sensitive to the value

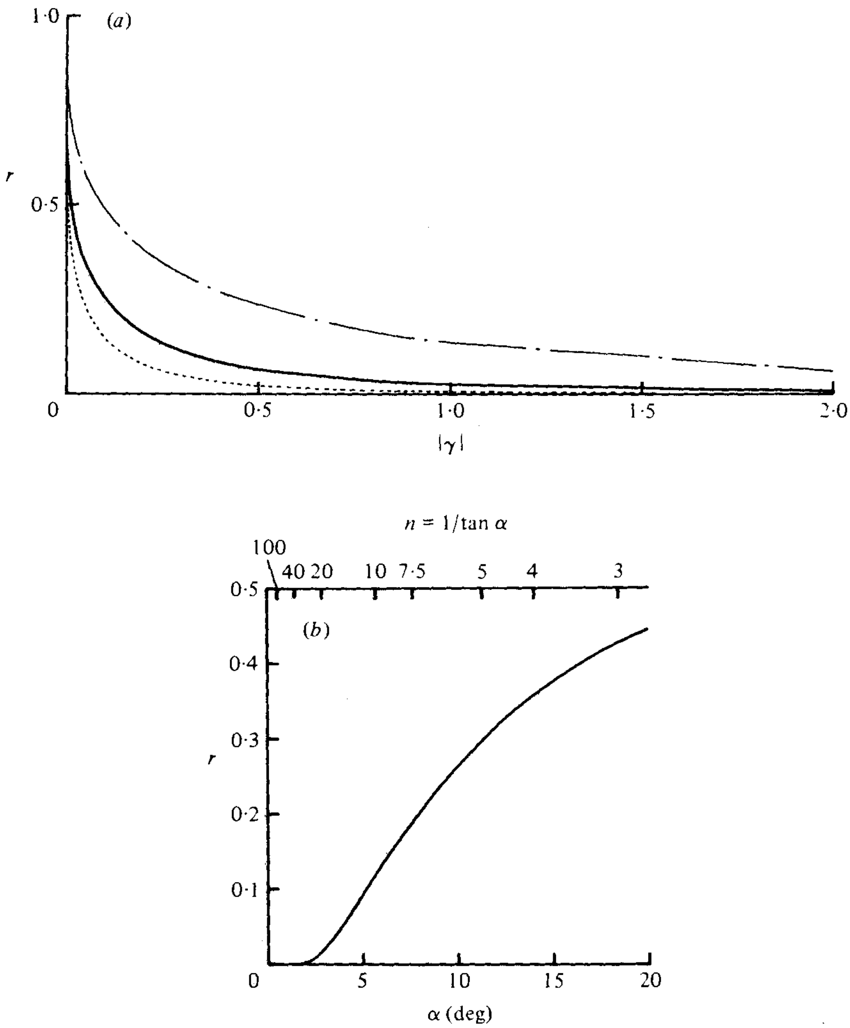

FIGURE 3. (a) The reflexion coefficient r as a function of the parameter $|\gamma|$, according to (4.12). ——, r_0; —·—, $r_{|\gamma|}$; ·····, $r_{-|\gamma|}$. (b) The dependence of r_0 on the beach slope for $\omega = 9{\cdot}06\ \mathrm{s}^{-1}$, $\nu = 0{\cdot}01\ \mathrm{cm}^2\ \mathrm{s}^{-1}$.

of Y, the phase of the reflected wave depends crucially on Y. Thus, the locations of the maxima and minima of $|\phi|$ provide a separate check on the appropriateness of the boundary conditions (cf. § 6).

The graphs in figure 3 (a) show the dependence of r on the parameter $|\gamma|$, from which it would appear that beaches having values of $|\gamma|$ in excess of about 0·25 are very efficient absorbers of energy. The graph in figure 3 (b) shows how r_0 depends on the beach slope for the particular values of ν and ω used in the experiments to be described.

A simple way of modifying the 'inshore' boundary condition and, in particular, the phase of the reflected wave is to impose the condition

$$\phi'(Y) = 0. \tag{4.13}$$

Such a condition corresponds to zero horizontal velocity at the position Y and could

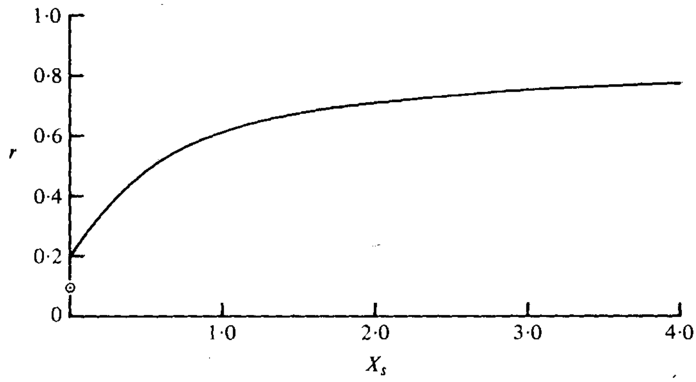

FIGURE 4. The reflexion coefficient for a plane beach terminated by a vertical wall as given by (4.14), with $\alpha = 0{\cdot}090$, $\omega = 9{\cdot}06\ \mathrm{s^{-1}}$, $\nu = 0{\cdot}01\ \mathrm{cm^2\,s^{-1}}$. ⊙ indicates the value for r_0 given by (4.12) for these conditions.

be achieved in practice by placing a vertical wall at a distance Y from the shore. (Of course, for the reasons outlined earlier, the 'virtual' position of the wall might not correspond exactly to its physical location.)

The reflexion coefficient arising from the condition (4.13) is

$$r_Y(\gamma) = \exp\left(|\gamma|(2/X_T)^{\frac{1}{2}}\right)\left|\frac{H_0^{(2)\prime}(\xi_Y)}{H_0^{(1)\prime}(\xi_Y)}\right|, \tag{4.14}$$

a graph of which is given in figure 4 for $|\gamma| = 0{\cdot}34$ and X_T infinite. Since this is the value of $|\gamma|$ to be used in our experiments it is interesting to compare values of r obtained from (4.14) and (4.12). For $Y = 0$, (4.12) gives a reflexion coefficient of 0·11 compared with a value of 0·19 obtained from (4.14), whereas with $Y = |\gamma|$ the respective values of r are 0·30 and 0·41. The latter number indicates that, even with a reflecting wall at $X = |\gamma|$, approximately 84 % of the incident wave energy would be dissipated on the beach.

Thus, when $|\gamma|$ is not too small, we see that the model equation (4.4) predicts that a large proportion of the incident wave energy will be absorbed in the boundary layer on the beach, irrespective of a wide variety of 'inshore' conditions. In this sense, the model appears to be fairly insensitive to the choice of the inshore boundary condition and we have decided therefore to fix on the boundary conditions (4.11) and (4.13) for the ensuing discussion.

5. Practical considerations

In seeking to test the results of these model calculations in the laboratory there are a number of experimental difficulties to be faced. As Ursell, Dean & Yu (1960) and Meyer & Taylor (1972) have observed, accurate measurements of reflexion coefficients can be rather tricky. One of the major problems is the establishment of a wavefield that is genuinely stationary over a period of time sufficient to allow a detailed set of measurements to be made. In addition, the relative levels of the 'incident' and 'reflected' components of the wavefield must be inferred from observations of the waves. For these inferences to be reasonably reliable it is necessary that the wave amplitudes be sufficiently small that nonlinear effects are negligible.

The usual arrangement for measuring reflexion coefficients is a uniform channel with a beach at one end and a wavemaker at the other. The normal practice is to drive the wavemaker with controls on the frequency that are tight enough for variations in the wavefield to be negligible. The amplitudes actually established in the wavefield are, for a given amplitude of the wavemaker, determined by the energy balances in the channel: the energy taken from the wavemaker, which must balance the total dissipation, depends on the wave amplitude at the paddle and the phase of these waves relative to the motion of the wavemaker.

These (and other) factors suggest that it is not worthwhile to attempt too accurate a description of the wavefield. So, in the uniform part of the channel, let us suppose that the flow field consists of two attenuating plane waves travelling in opposite directions, with the vertical displacement of the free surface being given by the real part of

$$\zeta = \zeta_0(e^{(\delta-ik)\tilde{x}} + r\,e^{-(\delta-ik)\tilde{x}+i\epsilon})\,e^{i\omega t}. \tag{5.1}$$

Here $\tilde{x}$ is the horizontal co-ordinate and all the parameters are real, with δ representing the decay of a wavemode along the uniform part of the channel, ϵ being a phase factor and k, ω denoting respectively the wavenumber and the frequency. If we take the origin for $\tilde{x}$ to be at the toe of the beach, then ζ_0 could be interpreted as an effective amplitude for the wave incident on the beach and r could be thought of as a 'reflexion coefficient' for the beach.

To determine r from measurements of the amplitude of the wave field we write $p = r\,e^{-2\delta\tilde{x}}$. Then the wave amplitude at a given station, is

$$|\zeta(\tilde{x})| = \zeta_0 e^{\delta\tilde{x}}[1+2p\cos(2k\tilde{x}+\epsilon)+p^2]^{\frac{1}{2}}. \tag{5.2}$$

If p is small we have that

$$|\zeta(\tilde{x})|/\zeta_0 \simeq e^{\delta\tilde{x}} + r\,e^{-\delta\tilde{x}}\cos(2k\tilde{x}+\epsilon) + O(r^2), \tag{5.3}$$

and so we can expect the graph of the wave amplitude to have an oscillatory component, at half the wavelength of the water waves, superimposed on a mean level that decays from the wavemaker to the beach at the rate δ.

For the particular case of $\delta = 0$ we have from (5.2) that $\sup|\zeta| = \zeta_0(1+r)$ and $\inf|\zeta| = \zeta_0(1-r)$, so that r can be determined as

$$r = \frac{\sup|\zeta| - \inf|\zeta|}{\sup|\zeta| + \inf|\zeta|}. \tag{5.4}$$

These formulae suggest that we extrapolate from the values of the local maxima and minima of $|\zeta|$ in the uniform part of the channel to estimate effective values at $\tilde{x} = 0$ and then use an expression of the form (5.4) to estimate r.

However, for a closed channel, the value of r thus defined may depend not only on the properties of the beach but on other factors, particularly the location of the wavemaker relative to the beach. When the absorption on the beach is large there will only be a small amount of wave energy reflected back to the wavemaker, and as this is not likely to affect the flow significantly it would appear that small values of r should be fairly reliably predicted by the above procedure. But it is possible that tuning effects for the complete system could give rise to anomalously large values of r, so for large values of r it may be necessary to consider the system as a whole.

5.1. *Experimental apparatus*

Since a considerable amount of experience had already been gained from other experiments with surface waves and also taking into account the above considerations, we decided to work with the following apparatus. A beach of slope $\alpha = 0{\cdot}090$ rad. was placed with its toe at a distance of 195·8 cm from an end of a uniform channel where there was a piston-type wavemaker. The bed of the channel had been carefully levelled up so that the still-water depth, except for the region over the beach, varied by no more than $\pm 0{\cdot}01$ cm from its mean value (though the actual water level, at a given station could be set to 0·001 cm through the use of a pointer gauge). For the experiments to be described below the mean depth was 3·00 cm. The channel was 30 cm wide.

The wavemaker was driven in an oscillatory manner by a synchronous motor forcing a crank attached to the paddle. When using very small amplitudes, namely about 0·01 cm of water wave, the waveforms of both the paddle and the water surface were very nearly sinusoidal. For all the experiments described below care was taken to ensure that the amplitudes were small enough for nonlinear effects not to be of any real importance for the motions in the uniform part of the channel.† The frequency for the driving voltage for the synchronous motor was derived from a crystal oscillator, which provided an extremely accurate frequency control, virtually eliminating phase drifting of the paddle. This was reflected in the observed steadiness of the wave field.

The conditions at the beach are important. For these experiments we used a plain sheet of perspex for the beach surface. The perspex was rubbed down with a very fine grade of emery paper, enabling the beach to retain a film of water well beyond the natural shoreline. This ensured even wetting at the shoreline and avoided the formation of a contact line at the shore, a feature we hoped might minimize the unknown influence of surface tension there (see § 6). Before the start of each experiment the surface of the water was skimmed with a vacuum pump. In order to modify the conditions at the shore some experiments were carried out with a vertical wall placed at various distances from the still-water line. This wall was a piece of square brass bar which spanned the channel and which had one face milled to the angle of the beach. The seal between the edges of the beach and the walls of the channel can be quite important: if it is not good, the wave-absorption properties of the beach can be changed significantly, as we shall indicate below.

The wave amplitudes were measured with a proximity gauge, a description of which is given in Barnard & Pritchard (1972). The principle on which the gauge works is that the capacitance between two plane electrodes can be related to the distance between them; the electronics associated with the instrument convert this relationship to a voltage which depends linearly on the distance between the electrodes, for a fairly large range of distances. The working range is determined, roughly, by the size of the electrodes. In the present case, one of the electrodes was the surface of the water and the other was basically a small plate (of diameter 1·1 cm, excluding its

† A detailed quantitative account of the importance of nonlinear effects in this experiment are given in Bona, Pritchard & Scott (1980).

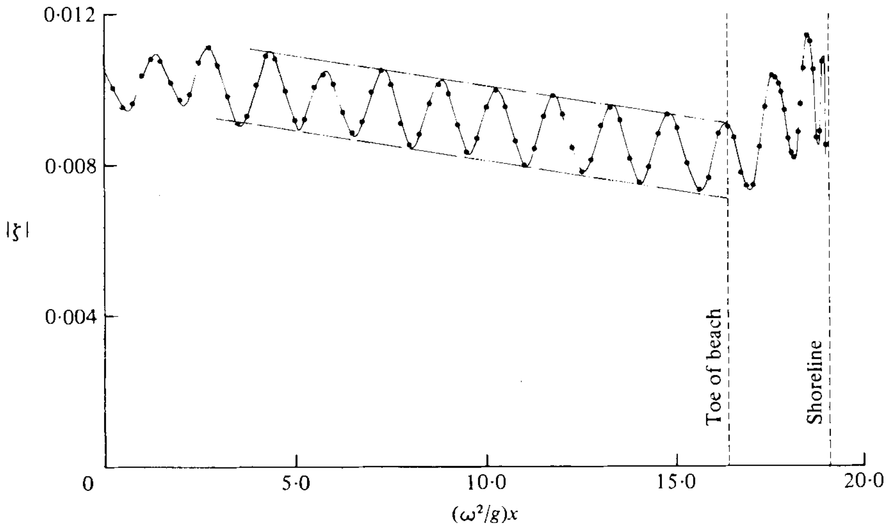

FIGURE 5. The observed wave amplitudes along the channel when $\omega = 9{\cdot}06\ \mathrm{s}^{-1}$, $\alpha = 0{\cdot}090$, $|\gamma| = 0{\cdot}34$.

'guard ring') clamped just above the free surface. This plate acted as a transducer to 'sense' the proximity of the water surface. The instrument is extremely versatile and will respond to frequencies up to at least 1 kHz. Accordingly we were able to make continuous recordings of the wave profile at various positions along the channel, from which the wave amplitude was determined. The wave amplitudes thus measured should have been accurate to better than 1 %.

We decided, for a number of reasons, to work with a wavelength-to-depth ratio of about 12:1. In the first place we had already established a considerable amount of experience under these conditions (e.g. see Bona, Pritchard & Scott 1980); but also the increased dissipative effects in shallow water help to offset the importance of any phase drifting of the wavemaker. Moreover, it is crucial that the wavelengths be larger than the channel width or the presence of transverse waves can be damaging (see the above-cited papers).

6. Experimental results

The wave profiles shown in figure 1, although having nonlinear characteristics near the shoreline, are very nearly sinusoidal at the larger distances from the shore. The wave forms in the uniform section of the channel were also very nearly sinusoidal in time, indicating that there the amplitude of the wave modes at the forced frequency should be given, to a very good approximation, by half the trough-to-crest height. For convenience we shall also use this measure in those cases (near the shore) where the waves have a noticeable harmonic content.

6.1. *The decay rate along the channel*

The results of a measurement of $|\zeta|$ along the channel and over the beach are given in figure 5. In this figure, and in the results to follow, the wave amplitudes are given as a proportion of the water depth in the uniform section of the channel. As anticipated in § 5, the measurements indicate a gradual decay in amplitude of the mode propagating towards the beach, together with an 'oscillatory' component to $|\zeta|$ arising from the reflected wave. The measurements over the beach are discussed below.

An empirical estimate of the rate-of-decay of the wavemode propagating toward the beach can be made via (5.3), giving a value of $1{\cdot}25\times 10^{-3}\ \text{cm}^{-1}$ for the data of figure 5. A theoretical estimate of the decay rate, made from boundary-layer arguments of the kind described in §§ 3, 4† (and see Hunt 1952) suggest a value of $7{\cdot}70\times 10^{-4}\ \text{cm}^{-1}$, which is about 40 % below the measured value. Since the free-surface boundary layer gives a decay rate of about $\frac{1}{50}$ of this value (see Landau & Lifschitz 1959) it would appear that either surface contamination, or the zone near the meniscus at the side walls of the channel played a significant role in determining δ (cf. Miles 1967). However, it should be noted that the decay rates are, in practice, so small that seemingly minor effects can be important‡ in determining the actual value of δ.

For the results shown in figure 5 the ratio of nonlinear to dispersive effects was about one. Thus, although there was very little evidence that higher harmonics were present in the wavefield in the uniform section of the channel, another experiment was made at amplitudes of about a fifth of those shown in figure 5. The decay rate and the reflected component were very nearly the same as those observed for the motions at the larger amplitudes.

6.2. *Reflexion coefficient*

Following the procedures outlined in § 5 we have used, as shown in figure 5, the local maxima and minima of $|\zeta|$ in the uniform part of the channel to estimate an 'effective' reflexion coefficient at the toe of the beach by means of a formula of the kind given in (5.4). For the data of figure 5 this leads to an estimate for r of 0·114 (indicating that the beach absorbed about 98·7 % of the wave energy incident upon it) and is typical of the results observed on different scales by other workers (see, for example, Greslou & Mahe 1955, Ursell *et al.* 1960). The reflexion coefficient r_0 predicted by (4.12) for the conditions of this experiment, is 0·106, which is remarkably close to the observed value. Indeed, such close agreement is probably somewhat fortuitous, as we shall indicate below, but it does suggest that this kind of theory can be used to describe the major features of the wave-absorption process in our experiments. In this experiment $|\gamma| = 0{\cdot}34$, corresponding to a horizontal distance of about 0·3 cm.

The comparison shown in figure 6 indicates that the theory predicts much larger wave amplitudes near the shore than those measured. However, for this comparison we have had to make measurements from wave profiles of the kind shown in figure 1: near the shore these were clearly influenced by nonlinear effects; also, with the

† In these calculations (based on the linear form of the free-surface boundary condition) the 'exact' dispersion relation and depth structure for the outer flow were used.

‡ For waves of sufficiently small amplitude the contact line of the water surface with the walls of the channel appeared not to move under the passage of the waves. But at larger amplitudes this is not always the case and there is evidence (see Barnard, Mahony & Pritchard 1977) to suggest that the damping rate of waves is not independent of the amplitude. In many of the present experiments the walls of the channel were lined with adhesive cotton bandage which was wet. In this case a contact line was not so easy to discern.

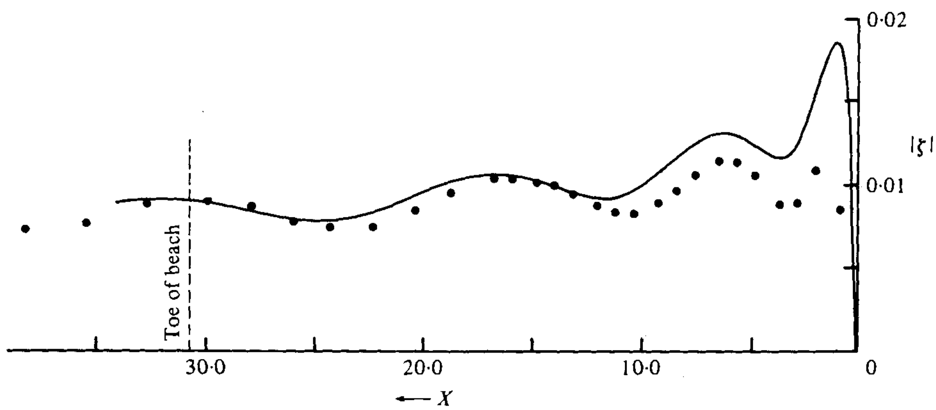

FIGURE 6. A comparison of the wave amplitudes over the beach with the theoretical amplitudes predicted by (4.8). The data are taken from figure 5.

'shorter wavelengths' encountered near the shore there may have been some inaccuracies introduced by the proximity gauge. The other feature of particular interest in figure 6 is the comparison of the positions of the nodes and antinodes of $|\zeta|$ and, in this respect, the theory appears to have predicted the empirical results very well.

The comparison made in figure 6 is an exacting test of the theory since the only arbitrariness in the theoretical solution is afforded by the specification of a wave amplitude, which has been chosen at the toe of the beach. It should, therefore, provide a good test for the choice of the boundary condition at the shore, since this has a strong influence on the locations of the maxima and the minima of $|\zeta|$, which are determined through the reflected wave component. Thus, the agreement here again suggested to us that the theory embodies the essential features of the absorption mechanism.

6.3. *Extraneous factors*

The reflexion coefficient from a beach can depend very sensitively on a number of features and we would now like to illustrate some of these factors, the first of which leads to a reduced reflexion coefficient and would appear to be a useful strategy to adopt in the design of beaches in the laboratory.

(*a*) *Leakage at the sides of the beach.* We have found that by deliberately allowing a leakage past the sides of the beach the reflexion coefficient can be influenced significantly. For example, with exactly the same conditions as those for the above experiment, except that a gap of 1 cm was left between the edge of the beach and the wall of the tank, the reflexion coefficient was reduced to 0·060.

(*b*) *Surface contamination.* With most laboratory experiments it is impossible to avoid a certain amount of surface contamination. Although the surface of the water was skimmed before each experiment, contaminants from both the water and the air will reach the surface during the course of an experiment and their importance can only be estimated from the 'repeatability' of the experiment. Fortunately this turned out to be very good if the measurements were made within an hour or so of skimming the surface. Moreover, owing to the very good control over the wavemaker, repetitions of the experiment over the course of a number of years gave virtually identical results to those quoted above.

	Experiment			
	1	2	3	4
$\omega/2\pi$ (Hz)	1·25	2·10	2·70	1·25
$\lvert\gamma\rvert$	0·198	0·431	0·628	0·198
r measured	0·33	0·30	0·08	0·23
r_0 predicted	0·16	0·07	0·05	0·16

TABLE 1. Reflexion coefficients for waves incident on a beach of angle 6·1°, as given by Feir (1966), are compared with the present theory.

While making some dye studies of the zone near the shoreline we inadvertently deposited a little surface-active material (the dye Gentian violet) on the water. This rendered the surface layers near the shore almost immobile and had a dramatic influence on the flow patterns near the shore.

(*c*) *The contact line on the beach.* The nature of the contact line of the water with the beach appears to be very important in determining the magnitude of the reflexion coefficient. The fact that, in our experiments, the beach had been roughened meant that a film of water could be retained well above the natural shoreline for a considerable period of time. The use of a 'wet' beach meant that the contact line was effectively eliminated and indeed the only simple and reliable way of detecting the position of the shoreline was to look for a small 'irregularity' in the reflected image of some object (usually the strip lighting) in the laboratory. These features made it difficult to be definitive about the movement of the shoreline under the action of the waves, but the overall impression was that there was very little, if any, movement of the shoreline. A beach constructed from expanded polystyrene, but which had also been roughened to retain a water film, gave almost identical results to those described above.

If, on the other hand, the water does not 'wet' the beach but, in the undisturbed state, has a definite contact angle at the shore, it would appear from some measurements of Feir (1966) that the reflexion coefficient can be affected significantly. Feir measured the reflexion coefficient as a function of wave steepness, but we consider here only the experiments at the smaller wave amplitudes. Four experiments are quoted, two of which were made under essentially the same conditions but yielded reflexion coefficients of about 0·23 and 0·33. Since the experiments were carried out with extreme care, this variability suggests just how important the conditions at the surface and the shoreline can be in determining the reflexion coefficient.

A summary of Feir's results, together with the predicted reflexion coefficients from the present theory (using the boundary condition (4.11)) are given in table 1. In this case the theoretical reflexion coefficients are quite a bit smaller than the measured values. One possible explanation for this is that, near the shore in the undisturbed state, there was a large deviation of the free surface from the horizontal. (It is interesting that the contact line at the beach seems to have remained fixed in these experiments, even though there was, under the action of the waves, considerable flexing of the free surface about this pivot.)† The wave amplitudes for Feir's experiments were, by and large, greater than those used in the present measurements.

† For Feir's measurements, the surface-tension parameter μ (see (4.7)) took the value 0·025 for experiments 1, 4 and the values 0·20 and 0·54 for experiments 2 and 3 respectively, suggesting that the surface-tension effects were not always unimportant.

For the measurements described here μ took a value of 0·063, so that surface tension was unlikely to have been very important for our experiments.

	Run no.							
	1	2	4	6	7	8	9	13
$\omega/2\pi$ (Hz)	0·310	0·595	1·087	1·299	1·266	1·163	1·087	0·901
$\|\gamma\|$	0·060	0·159	0·393	0·513	0·493	0·434	0·393	0·296
r measured	0·041	0·042	0·021	0·043	0·039	0·069	0·059	0·024
r_0 predicted	0·323	0·187	0·085	0·065	0·067	0·077	0·085	0·112
	Run no.							
	14	15	16	17	18	19	21	22
$\omega/2\pi$ (Hz)	0·613	0·478	0·272	0·787	1·075	0·800	1·266	1·176
$\|\gamma\|$	0·167	0·115	0·049	0·242	0·386	0·248	0·493	0·442
r measured	0·021	0·057	0·451	0·036	0·057	0·033	0·022	0·036
r_0 predicted	0·183	0·230	0·350	0·134	0·086	0·132	0·067	0·075
	Run no.							
	23	24						
$\omega/2\pi$ (Hz)	1·053	1·042						
$\|\gamma\|$	0·374	0·368						
r measured	0·054	0·055						
r_0 predicted	0·088	0·090						

TABLE 2. Reflexion coefficients from Ursell *et al.* (1960) for a beach of slope $\alpha = 0{\cdot}0681$, compared with the present theory. Note that the beach allowed seepage past its edges.

6.4. *Other experiments*

A very careful set of measurements of wave reflexion from a beach was made by Ursell, Dean & Yu (1960). In these experiments the beach was of a plane, impermeable, varnished material, with a small gap of about 0·6 cm between the beach edges and the walls of the flume, allowing seepage past the edge of the beach. Since it is not clear whether or not breaking took place over the beach, comparisons with this work should be treated with some caution. Nevertheless we felt they should be made. Listed in table 2 are the results given by Ursell *et al.* (using their numbering for the experiments) together with the reflexion coefficient predicted by (4.12), taking $X_{T'}$ to be infinite.

Apart from runs 1, 2, 14, 15, the agreement between the measured and predicted values of r is close enough (especially since the effect of seepage should be allowed for) to suggest that a good proportion of the wave absorption occurred through the effects of bottom friction. The four predictions giving poor agreement with the measured values were for experiments at much larger wave periods, giving smaller values of $|\gamma|$ than the other runs; that is, with the exception of run 16, which yielded an anomalously large reflexion coefficient, and was reasonably well predicted by the theory.

A set of measurements of reflexion coefficients for waves covering a fairly large range of amplitudes has been reported by Greslou & Mahe (1955) (and see also Meyer & Taylor 1972). Unfortunately complete information regarding the details of the experiments is not given, but the suggestion from their paper is that the reflexion coefficient was essentially independent of the frequency for the range of frequencies covered in the experiments. Indeed, from the scatter among the data given in the paper, it would appear that the reflexion coefficients were not determined to an accuracy of better than about $\pm 5\,\%$ and, given this variability, the effects of changing the frequency

	α			
	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{5}$
r_0 predicted $T = 0{\cdot}9$ s	0·01	0·04	0·16	0·35
r_0 predicted $T = 1{\cdot}6$ s	0·04	0·11	0·27	0·55
r measured	0·04	0·07	0·14	0·44

TABLE 3. Reflexion coefficients measured by Greslou & Mahe (1955) for beaches of various slopes. The theoretical values are given for periods at the ends of the range covered by the experiments.

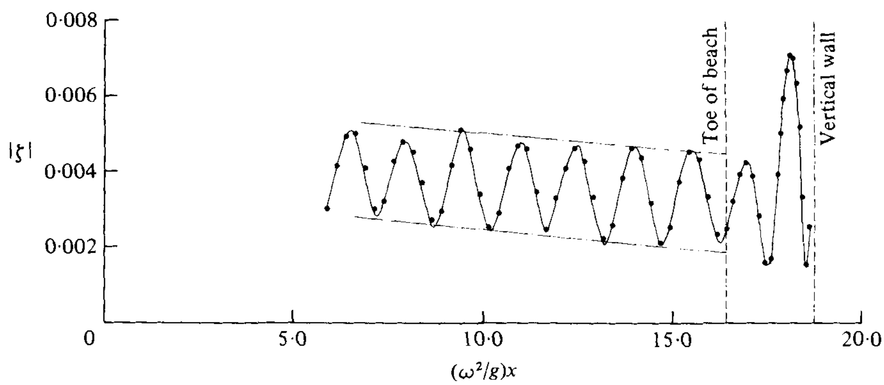

FIGURE 7. The wave amplitudes observed in the main part of the channel with the same operating conditions as for the results of figure 5, except that the beach was terminated by a vertical barrier at a distance of 2·0 cm from the 'natural' shoreline.

were probably not noticeable. Nevertheless, the overall shapes of the graphs for the observed reflexion coefficients, as a function of the beach slope, are similar to the one given in figure 3 (*b*). Taking the reflexion coefficients observed for the smallest waves used in their experiments, we have compared the results with the present theory, by calculating the reflexion coefficients for the smallest and largest frequencies at which the measurements were made. The results of this comparison are given in table 3, and again the agreement is seen to be very good over the entire range.

6.5. *A plane beach terminated by a vertical wall*

Shown in figure 7 are the results of a measurement of the wave amplitudes observed when the beach was terminated by a vertical cliff at a distance $y = 2{\cdot}0$ cm from the 'natural' shoreline. The estimate of the reflexion coefficient made, as above, by extrapolating from the wavefield in the uniform section to the toe of the beach is $r = 0{\cdot}40$, which is significantly below the theoretical value of 0·69 given by (4.14). The reason for this discrepancy may depend on a number of factors. For example, in the above experiment, the end wall had a layer of adhesive bandage on it so that it was essentially wet and the meniscus contact angle should have been very small. With no bandage attached to the cliff, but the conditions otherwise unchanged, we observed a reflexion coefficient of 0·44. Also it appears that nonlinear effects might have been important in this experiment. The wave amplitudes near the cliff were almost twice as large as those anywhere else in the channel and it is possible that this feature, coupled with the meniscus at the wall and the fairly small depths there

	Experiment				
	A	B*	C	D	E
Position of wall y (cm)	1·4	2·0	2·5	4·0	4·5
r measured	0·33	0·40	0·35	0·45	0·57
r predicted	0·65	0·69	0·72	0·77	0·78

TABLE 4. Some measurements of the reflexion coefficient with a vertical wall terminating the beach, and the theoretical predictions given by (4.14), for $\omega = 9{\cdot}0666\ \mathrm{s}^{-1}$. Experiment (*), as described in the text, was made at a smaller amplitude than the others.

(approximately 2 mm), meant that the boundary condition (4.13) did not provide a good model for the actual flow. The dependence of the reflexion coefficient on the amplitude level of the wavefield is illustrated by the results given in table 4. These measurements, with the exception of experiment B, were made at the outset of this study and had mean wave amplitudes at the toe of the beach of 0·008. It is interesting that, with the smaller wave amplitudes used in experiment B, much larger reflexion coefficients were observed than those in experiments A and C.†

As both the theory and the results of table 4 suggest, the absorption properties of the beach rapidly diminished as the cliff was moved out from the natural shoreline. Moreover, it would appear from all the data that, on these scales, the actual conditions at the shore can be very important in determining the reflexion coefficient in a given experiment.

6.6. *Dye studies*

It was often found in our experiments, when some crystals of potassium permanganate were dropped onto the beach, that the resulting dye patterns showed some curious properties. An example is given in the photograph shown in figure 8 (plate 1). The study revealed quite a marked drift of dye from crystals lying on the bottom and in the zone near the shoreline. Some examples of this drift can be seen in the photograph, the dark spots being permanganate crystals on the bottom.‡ This dye moved rapidly up the beach to a zone about $2\frac{1}{2}$ cm from the shoreline (which appears in the photograph as a faint horizontal line)§ and then drifted away from the shoreline again at a higher level. The drift from crystals placed fairly close to the shoreline consisted of a flux down the beach and out to the deeper waters, and the overall picture that emerged after some time is that shown in the photograph. Of course this is only

† With the large reflected wave components prevailing in these experiments, we also considered the possibility discussed in § 4 that, in order to make reliable comparisons, the system should really be treated as a whole. This can be done by patching together, at the toe of the beach, two solutions of the form (4.8) (except that, in the uniform section of the tank, we can expect exponential functions rather than Hankel functions). We have done this patching by assuming that the wave amplitude is continuous and that the mass flux across the vertical plane through the toe of the beach is continuous. (The conditions effectively correspond to choosing ϕ, ϕ_x to be continuous.) At the wavemaker we assume that ϕ_x is some specified function of z. Carrying through this procedure when $y = 1{\cdot}4$ cm, it was found that r should be modified to 0·59 from the value 0·65 calculated from (4.14). But this is not nearly large enough a correction to account for the differences shown in table 4.

‡ Near the free surface, the expected drift towards the shore was evident.

§ There is actually a double image of the shoreline present on the photograph, the second image being a reflexion of the (illuminated) shoreline in the underside of the beach.

a very superficial picture of the dye movements which, on closer study, revealed considerable complexity. For example, the distinct edge formed by the dye patch in this experiment was much more diffuse when the incident wave amplitude was decreased, suggesting that the feature shown here was a result of nonlinearities in the region near the shore. Also, in a vertical plane, there was evidence of a cellular pattern to the dye motions, but this was too weak to show up in the photograph.

7. Discussion

Our main aim in undertaking this work was to try to isolate some of the mechanisms that might be important in absorbing wave energy on beaches. When waves break on ocean beaches it is not clear just how much of the wave energy has already been dissipated on the ocean bed, but in some of our laboratory experiments the evidence points to this being the main means by which the wave energy was degraded. For this reason we wanted, if possible, to develop a fairly consistent theory for the laboratory experiments in the hope that this may provide some guidance in coming to terms with the oceanographic problem. In the laboratory the comparisons between the theory and a wide range of experiments appears to be quite good; however, our main concern here was to see whether or not the theory predicted the right *scale* on which the absorption takes place, rather than worrying too much about the detailed agreement. The reason was that a number of uncertainties might have obscured the picture: for example, it is difficult to estimate the part played by the contact line at the shore, especially if it should move, and certainly Feir's (1966) work indicates that this can be a very delicate issue (although this particular problem appears to have been obviated in the present experiments). But, in addition, the appropriate boundary condition to choose at the shoreline, on the basis of a boundary-layer calculation over the beach, must be a matter of guesswork. Thus, the fact that the theory appears to predict fairly well the experimental results for a plane beach suggests to us that the theory embodies the essential mechanism for wave absorption in these experiments. In this respect it was very interesting to us that the same theory failed badly in predicting the reflexion coefficient when the beach was terminated by a vertical barrier a few centimetres off shore, thereby suggesting either that the boundary condition at the barrier was not appropriate or that nonlinearities were an important contributing factor in this instance.

The theory of § 4 suggests that the parameter $|\gamma| = (\nu\omega^3)^{\frac{1}{2}}/g\alpha^2$ should determine a length scale on which we can expect frictional effects on the bottom to be important. For the experiments relating to figures 5 and 6, $|\gamma|$ was approximately 0·34, but it is also instructive to examine its size for a typical ocean beach. Here the situation is complicated by the choice of a suitable eddy viscosity for a turbulent flow over a rough bed with moving sediment, however, the work of Jonsson & Carlsen (1975) suggests that a value for ν of 10 cm^2 s^{-1} is likely to underestimate the eddy viscosity on ocean beaches. Thus, for 10 s waves incident on a beach of slope 0·01 it follows that $|\gamma|$ should take a value of say 16, or more. Such large values suggest that the asymptotic structure of (4.8) for large $|\xi|$ can be used, in which case the viscous effects should dominate the shelving effects when $X < 2|\gamma|^2$. This corresponds, in the present example, to a distance of about 125 m, whereas $|\gamma|$ corresponds to a distance of only 4 m. Since the breaking zone is usually well within 125 m, this would suggest that most wave breaking arises because of local variations in the beach slope near the shore, or through the presence of an off-shore bar.

Certainly it would appear from the results of this study, that the boundary layer at the bottom plays a significant role in determining the overall energy balances for waves incident on a beach.

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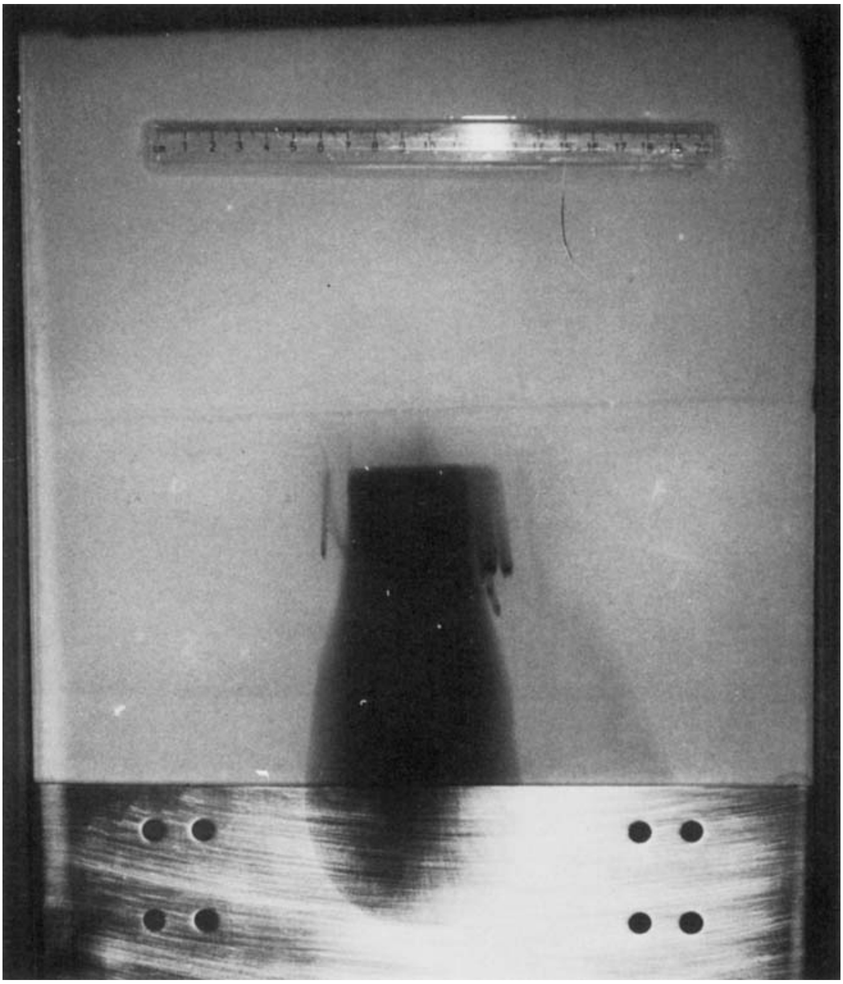

FIGURE 8. The result of dye movement after some crystals of potassium permanganate had been dropped onto the beach. Note that waves were being generated at the time this photograph was taken.